Path integral regularization of pure Yang-Mills theory

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Abstract

In enlarging the field content of pure Yang-Mills theory to a cutoff dependent matrix valued complex scalar field, we construct a vectorial operator, which is by definition invariant with respect to the gauge transformation of the Yang-Mills field and with respect to a Stueckelberg type gauge transformation of the scalar field. This invariant operator converges to the original Yang-Mills field as the cutoff goes to infinity. With the help of cutoff functions, we construct with this invariant a regularized action for the pure Yang-Mills theory. In order to be able to define both the gauge and scalar fields kinetic terms, other invariant terms are added to the action. Since the scalar fields flat measure is invariant under the Stueckelberg type gauge transformation, we obtain a regularized gauge-invariant path integral for pure Yang-Mills theory that is mathematically well defined. Moreover, the regularized Ward-Takahashi identities describing the dynamics of the gauge fields are exactly the same as the formal Ward-Takahashi identities of the unregularized theory.

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1 Introduction

To understand the dynamic of pure Yang-Mills (YM) theory [1], in all the range of the energy scale, one needs first a gauge-invariant regularization in four-dimensional spacetime. In the seventies, Wilson has built a gauge-invariant regularization of YM theory by approximating the continuous four-dimensional spacetime as a discrete lattice [2]. In this approach gauge invariance is obvious, but the continuous symmetries of the physical spacetime, such as Lorentz invariance are clearly lost. The continuous YM action is recovered when the lattice spacing goes to zero. In this regime the theory is perturbative, and for each physical quantity, the result calculated in lattice gauge theory must match the result given in a perturbative continuous regularization, such as dimensional regularization [3]. For instance, as the lattice spacing goes to zero, some integrals which arise from the diagrammatic expansion of the action of YM lattice perturbation theory, like the tadpole integrals, become not regularized in the infrared (IR) and need an intermediate regularization [4, 5]. Therefore, since the matching between the discrete and continuous regularization is really not obvious, for the study of the evolution of physical quantities with the energy scale, ranging from the IR to the ultraviolet (UV) domains, a continuous nonperturbative regularization in four-dimensions which will preserve both the spacetime symmetries and the gauge invariance of the theory, will be very appealing.

Apart from dimensional regularization, which is in fact purely perturbative, two different classes of continuous regularization were essentially developed as a starting point for a nonperturbative approach, since the advent of lattice regularization. 1)In the method of higher covariant derivative supplemented by the introduction of Pauli-Villars (PV) regulator fields [1, 6], which are necessary to regularize the one loop diagrams, it is the action which is regularized. Because of Gribov [7] ambiguities, in this scheme the regularized path integral has nevertheless only a perturbative meaning. 2)In the method of stochastic regularization [8], by construction it is not the YM action which is regularized with the help of covariant derivative, but the second order Schwinger-Dyson (DS) equations, which are obtained after integrating out the Gaussian noise field. Since in this scheme the quantization occurs without Faddeev-Popov ghosts, the regularized DS equations can in principle be used to study the nonperturbative aspects of YM theory. However in this scheme the renormalization program is not straightforward. In fact, due to the lack of a regularized action, the regularized Ward-Takahashi (WT) identities can not be deduced from the onset, and one must assume [9] that they are identical to the unregularized ones.

In order to be able to apply exact renormalization group techniques [10], we require that the regularization scheme does not work only on Feynman diagrams, or DS equations, but

regularize the YM path integral as a whole, explicitly and not only formally[1]. One method to regularize the path integral of a given theory, is to add a cutoff function in the quadratic term of the action, in order that the propagator becomes a rapidly decreasing function of the square of the momenta in momentum space. This idea is one of the cornerstones of the construction of regularized path integrals which are suited for the derivation of exact renormalization group techniques [11]. In this approach, whereas the path integral that quantizises the theory is indeed well defined, the regularization breaks explicitly the internal local symmetries of the original action. This is because the gauge transformation mixes different scales of the gauge field momenta. Only recently was it shown that this drawback can be circumvented by enlarging the fields content of the theory : The higher covariant derivative regularization is applied to a spontaneously broken supergroup which embeds the original physical non-Abelian gauge group [12]. The original YM theory is then regularized by higher covariant derivatives supplemented by a finite set of PV regulator fields, if some preregularization is used [13]. But, because the PV regulator fields introduced in the action have negative norm by definition [14], the path integral of the theory is only defined formally and perturbatively.

This paper shows that the path integral of $SU(3)$ YM theory can be regularized as a whole, in a continuous and gauge-invariant way, by enlarging the field content of the theory by a finite set of complex scalar fields, smeared with cutoff functions. In order to do so, we will follow the lines of reasoning which have allowed us to regularize the path integral of QED [15] in a gauge-invariant manner. The action is constructed in its essence with a gauge-invariant building block $\mathcal{A}_\mu^{Inv}$. This vectorial operator of dimension one is a function of the gauge field and of the smeared complex scalar fields, and converges to the standard gauge field A^μ when the cutoff goes to infinity. The variation of this vectorial operator under a local gauge transformation of the gauge field is compensated by a group transformation of the smeared complex scalar field which depends on the cutoff scale and which plays a role similar to that of Stueckelberg compensating fields [16]. The regularization is implemented in the action by smearing these invariants by smooth cutoff functions. Then all interaction vertices of the regularized action are smeared with a cutoff function and the quadratic terms in the fields can be inverted to give rise to gauge and scalar fields propagators which are rapidly decreasing functions of the square of the momenta. Since the measure and the regularized action is invariant under both the gauge transformations of the vectorial and scalar fields, when the integration over the scalar fields is being assumed implicitly, the regularized WT identities relative to the gauge field are exactly the same as those which can be deduced naively from the unregularized action without gauge fixing.

The paper is organized as follows. Section II is devoted to the explicit construction of the

[1]For instance in the case of dimensional regularization the path integral has only a perturbative meaning.

regularized gauge-invariant action of the YM theory with the help of the invariant vectorial operator $\mathcal{A}_\mu^{Inv}$. The flat measure relative to the scalar fields is showed to be invariant under the gauge Stueckelberg type transformation. In Section III, in order to be able to invert the gauge vector kinetic term without breaking explicitly the whole gauge invariance of the regularized action, we add to the action a gauge-invariant part which is built with the invariant operator $\mathcal{A}_\mu^{Inv}$ and which is reminiscent of the standard covariant gauge fixing term. Here the addition of a mass term for the real and imaginary parts of the complex scalar fields is required, so that the full action remains regularized. In Section IV, we show that the mass term relative to the real part of the complex scalar field, that is not invariant under the Stueckelberg type of gauge transformation, is in fact harmless. This implies that, after integration on the scalar fields, the regularized WT identities describing the dynamics of the YM gauge fields are exactly the same as the formal WT identities of the unregularized theory without gauge fixing.

2 The regularized gauge-invariant action

If $A_\mu = A_\alpha^\mu T^\alpha$, is the $SU(3)$ color gauge vector boson, we define the matrix valued gauge-invariant operator

$$\mathcal{A}_\mu^{Inv} = U^+ A_\mu U - \tfrac{1}{2}(U^+\partial_\mu U - \partial_\mu U^+ U), \tag{2.1}$$

with the generators T^α related to the Gell-Mann matrices by

$$T^\alpha = i\tfrac{\lambda_\alpha}{2}. \tag{2.2}$$

Here, U is a 3×3 dimensionless complex matrix acting on the generators T^α, and hereafter an expression like $\partial_\mu AB$ means that the partial derivative acts only on the factor A. By construction the vector $\mathcal{A}_\mu^{Inv}$ is anti-Hermitic and invariant under the local transformation of A^μ and U by an arbitrary $SU(3)$ matrix

$$A_\mu \quad \rightarrow \quad V A_\mu V^+ + \partial_\mu V V^+ \tag{2.3}$$

$$U \quad \rightarrow \quad VU. \tag{2.4}$$

The invariance of A_μ^{Inv} relies only on the property of unitarity of the matrix V. Thus, the unitarity [2] of the matrix U is in general not required. In order to construct a regularized gauge-invariant action, we start with the gauge-invariant expression

$$F_{\mu\nu}^{Inv}(z) = \int dx \; \left\{\rho_1^{(-1)}(z,x)\left(\partial_\mu A_\nu^{Inv} - \partial_\nu A_\mu^{Inv}\right) - \rho_2(z,x)[A_\mu^{Inv}, A_\nu^{Inv}]\right\}. \tag{2.5}$$

[2]If U were unitary, the definition (2.1) is that of the gauge-invariant operator given in [17].

Here and hereafter we work in Minkowski space. We choose the signature of the metric to be $(1,-1,-1,-1)$ and the notation $dx \equiv d^4x$ and $d\bar{k} \equiv d^4k/(2\pi)^4$. In the expression (2.5) the ultraviolet UV or infrared IR regularization are implemented through the real scalar functions

$$\rho_i(x,y) \;=\; \int d\bar{k}\; e^{-ik(x-y)}\rho_i(\tfrac{k^2}{\Lambda^2},\kappa) \tag{2.6}$$

$$\rho_i^{(-n)}(x,y) \;=\; \int d\bar{k}\; \frac{e^{-ik(x-y)}}{\rho_i^n(\frac{k^2}{\Lambda^2},\kappa)\;+\;\eta\rho_i^{-n}(\frac{k^2}{\Lambda^2},\kappa)}, \tag{2.7}$$

κ and Λ being respectively the IR and UV cutoff scales. The term proportional to the function $\rho_1^{(-1)}$ will contribute to the kinetic part of the action, and hence to the inverse of the propagator. As a result, when the path integral of the theory is expressed in momentum space, the infinitesimal real parameter η, which was only introduced to give a mathematical meaning to the Fourier transform (2.7), can be set to zero . In Euclidean momentum space, the cutoff functions $\rho_i(k)$ are rapidly decreasing functions of k^2 in both the IR and UV domains and verify the condition [3]

$$\lim_{\Lambda\to\infty,\kappa\to 0}\rho_i(k,\Lambda,\kappa) \;=\; 1. \tag{2.8}$$

By definition, the functions (2.6) are regularized forms of Dirac's δ function, i.e.

$$\lim_{\Lambda\to\infty,\kappa\to 0}\rho_i(x,y) \;=\; \delta(x-y). \tag{2.9}$$

The expression (2.5) is apparently quartic in the matrix U. In order to simplify and to express the operator (2.5) in terms of the usual YM stress tensor, we suppose as a first step, that the matrix U is unitary. Then using the relation

$$\partial_\mu U^+ U = -U^+\partial_\mu U \tag{2.10}$$

(2.5) becomes

$$\begin{aligned} F_{\mu\nu}^{Inv}(z) \;=\; \int dx\, \Big\{ & U^+F_{\mu\nu}^{Reg}(z,x)U + (\rho_1^{(-1)}(z,x)-\rho_2(z,x))\Big[(\partial_\mu U^+A_\nu U + U^+A_\nu\partial_\mu U \\ & -\partial_\mu U^+\partial_\nu U) - (\mu\leftrightarrow\nu)\Big]\Big\}, \end{aligned} \tag{2.11}$$

where the regularized YM stress tensor is defined by the expression

$$F_{\mu\nu}^{Reg}(z,x) = \rho_1^{(-1)}(z,x)\,(\partial_\mu A_\nu - \partial_\nu A_\mu) - \rho_2(z,x)[A_\mu,A_\nu]. \tag{2.12}$$

[3] Notice that there are a large number of cutoff functions which are valid in the IR and UV domains and which converge to the Dirac's δ function as the cutoff Λ goes to infinity, whatever the value of κ. For instance, this is the case for the cutoff function $\frac{k^2}{k^2+\frac{\kappa^4}{\Lambda^2}}e^{\frac{k^2}{\Lambda^2}}$.

If one relaxes the condition of unitarity of the matrix U, one can check that the new expression (2.11) of the operator (2.5) remains invariant under both transformations (2.3) and (2.4). Notice that in this case the relation (2.10) does not hold true, and hence the relations (2.5) and (2.11) define two distinct invariants. We select the last expression (2.11) as the building block of the regularized YM action. In the limit $\Lambda \to +\infty$, $\kappa \to 0$, one can set the parameter η to zero, and the right-hand side of (2.11) converges to the operator $U^{+} F_{\mu\nu} U$. The trace of the operator $F^{Inv}_{\mu\nu} F^{Inv\mu\nu}$ does converge to the standard, unregularized Lagrangian density of QCD if the matrix U is unitary.

The next step consists in choosing the form of the dimensionless matrix U. Since the matrix U is needed to restore the gauge invariance in (2.11), it can be expressed as a functional of some scalar fields which will play the role of Stueckelberg compensating fields [16, 17].

First to avoid any nonpolynomial interaction between the Stueckelberg fields, which will enter the action we are looking for, second to have the path integral measure invariant under the transformation of the Stueckelberg fields that induces the transformation (2.4), and third, so that the invariant operator $\mathcal{A}^{Inv}_{\mu}$ converges to the standard gauge vector A_μ as the cutoff Λ goes to infinity, we define the 3×3 dimensionless matrix U as the linear combination

$$U = I + \tfrac{1}{\Lambda}\phi. \tag{2.13}$$

Here, the matrix ϕ is expressed in terms of the complex scalar fields ϕ_a as

$$\phi = \phi_a T^a, \tag{2.14}$$

and by convention, we use Latin letters when the index runs from 0 to 8. In the definition (2.14), except the matrix T^0 which is given in term of the identity matrix I as

$$T^0 = \tfrac{i}{\sqrt{6}} I, \tag{2.15}$$

the eight remaining matrices T^α are defined in (2.2). With the definition (2.15), the matrices T^a are normalized as

$$\mathrm{Tr}(T^a T^b) = -\tfrac{1}{2}\delta_{ab}, \tag{2.16}$$

and the properties of the matrices (2.2) are recalled in Appendix A. We do not impose any constraint on the nine complex fields ϕ_i, so that the matrix U is not unitary in general[4]. Due to its definition, this matrix is an element of the Lie algebra of the $U(3)$ unitary group. Since the matrix U transforms like (2.4), the transformed matrix ϕ' verifies the equation

$$\phi' = V\phi + \Lambda(V - I). \tag{2.17}$$

[4]For a unitary matrix U, the invariant measure of the fields ϕ_i is hard to define, as it is the case in the nonlinear sigma model [14].

It is clear that the set of transformations (2.17), acting on the matrix ϕ (2.14), is a nonlinear representation of the $SU(3)$ group. In fact, if g is the group element of $SU(3)$ represented by the matrix V, the transformed field ϕ' can be written as

$$\phi' \;=\; {}^{g}\phi, \tag{2.18}$$

and we have

$$^{g'}({}^{g}\phi) \;=\; {}^{g'g}\phi, \tag{2.19}$$

with the identity group element e represented by the identity matrix I, and the inverse group element g^{-1} represented by the matrix V^{-1}.

How does the measure of the scalar field transform under the transformation (2.17) ? As show in Appendix B, if Φ is the nine-dimensional complex vector of components ϕ_i, the transformed fields ϕ_i' are given according to Φ by the vectorial relation

$$\Phi' = S^{-1}T\,(V)\,S\Phi + \Lambda S^{-1}T\,(V-I)\,C_I. \tag{2.20}$$

Here the nine-dimensional matrix representation $T(V)$ of the $SU(3)$ gauge group and the constant vector C_I are respectively defined in (B.1) and (B.2), and S is the constant matrix defined by the relation (B.6).

Then the Jacobian of the transformation (2.20) is given by

$$\left|\tfrac{\delta\phi_i'(x)}{\delta\phi_j(x')}\right| = \delta(x-x')\det\left(S^{-1}T\,(V)\,S\right) = \delta(x-x'), \tag{2.21}$$

because the determinant of $T\,(V)$ is unity (B.4). In the same manner, one can see that the Jacobian of the transformation of the Hermitic conjugate scalar fields ϕ_i^+, induced by the transformation (2.17), is also unity. As a result, since the scalar fields ϕ_i and ϕ_i^+ are independent variables the measure

$$\mathcal{D}\phi\mathcal{D}\phi^+ \equiv \prod_{i=0}^{i=8}\mathcal{D}\phi_i\prod_{i=0}^{i=8}\mathcal{D}\phi_i^+, \tag{2.22}$$

is invariant under the transformation (2.17).

Since the form of the matrix U is now given by the expression (2.13), the invariant tensor (2.11) is recast in terms of the regularized YM stress tensor (2.12) in the expression

$$F_{\mu\nu}^{Inv}(z) = \int dx\;\left\{F_{\mu\nu}^{Reg}(z,x) + \rho_2\,(z,x)\,G_{\mu\nu} + \left(\rho_1^{(-1)}\,(z,x) - \rho_2\,(z,x)\right)H_{\mu\nu}\right\}, \tag{2.23}$$

where the antisymmetric tensors $G_{\mu\nu}$ and $H_{\mu\nu}$ are given in terms of the gluons fields A_μ, the standard YM strength tensor $F_{\mu\nu}$ and the complex scalar fields ϕ by

$$\begin{aligned} G_{\mu\nu} &= \tfrac{1}{\Lambda}(\phi^+F_{\mu\nu} + F_{\mu\nu}\phi) + \tfrac{1}{\Lambda^2}\phi^+F_{\mu\nu}\phi & (2.24)\\ H_{\mu\nu} &= \left[\tfrac{1}{\Lambda}\partial_\mu(\phi^+A_\nu + A_\nu\phi) + \tfrac{1}{\Lambda^2}\partial_\mu(\phi^+A_\nu\phi) - \tfrac{1}{\Lambda^2}\partial_\mu\phi^+\partial_\nu\phi\right] - \left[\mu\leftrightarrow\nu\right]. & (2.25)\end{aligned}$$

A candidate for a regularized form of the YM action can be defined as

$$S(A,\phi^+,\phi)=\tfrac{1}{2g^2}\,\mathrm{Tr}\int dz F^{Inv\ \mu\nu}(z)F^{Inv}_{\mu\nu}(z), \tag{2.26}$$

g being the dimensionless strong coupling constant. The explicit form of this action can be expressed as

$$S(A,\phi^+,\phi)=S_{YM}+S_{A\phi^+\phi}, \tag{2.27}$$

where the pure gauge part of this action is given in terms of the regularized YM stress tensor (2.12) by the expression

$$S_{YM}=\tfrac{1}{2g^2}\,\mathrm{Tr}\int dxdydz\ F^{Reg\ \mu\nu}(z,x)F^{Reg}_{\mu\nu}(z,y), \tag{2.28}$$

and where $S_{A\phi^+\phi}$ is given in terms of the operators $G_{\mu\nu}$ (2.24), $H_{\mu\nu}$ (2.25) and $F^{Reg}_{\mu\nu}$ (2.12) by,

$$\begin{aligned}S_{A\phi^+\phi} \;=\;& \tfrac{1}{2g^2}\,\mathrm{Tr}\int dxdydz\ \Big\{\rho_2(z,x)\rho_2(z,y)G^{\mu\nu}G_{\mu\nu}(y)\\
&+(\rho_1^{(-1)}(z,x)-\rho_2(z,x))(\rho_1^{(-1)}(z,y)-\rho_2(z,y))H^{\mu\nu}H_{\mu\nu}(y)\\
&+2\Big[\rho_2(z,x)G^{\mu\nu}F^{Reg}_{\mu\nu}(z,y)+(\rho_1^{(-1)}(z,x)-\rho_2(z,x))H^{\mu\nu}F^{Reg}_{\mu\nu}(z,y)\\
&+\rho_2(z,x)(\rho_1^{(-1)}(z,y)-\rho_2(z,y))G^{\mu\nu}H_{\mu\nu}(y)\Big]\Big\}.\end{aligned} \tag{2.29}$$

Under what conditions is the action (2.27) regularized ? In order to study this point, we rewrite the action (2.29) as a sum of five terms

$$S_{A\phi^+\phi}=S_{GG}+S_{HH}+S_{GF}+S_{HF}+S_{GH}, \tag{2.30}$$

each term being respectively associated to the product of the operators which enter the expression of the action (2.29) from the left to the right. The Fourier transform of these terms are given in Appendix C. First of all, we suppose that the cutoff functions ρ_1 and ρ_2 (2.6) are chosen in such a way that the product $\rho_1^{-1}\rho_2$ are still rapidly decreasing functions of k^2 in momentum Euclidean space. The simplest choice is

$$\rho_2(k)\sim\rho_1^2(k). \tag{2.31}$$

Then, since the kinetic term arising from the term $F^{Reg}_{\mu\nu}$ (2.28) and (2.12) generates a propagator [5] proportional to ρ_1^2 in momentum space, all the terms of the action (2.27) are regularized with respect to the gauge field, except the terms of (2.29) which do not contain the factor

[5] In the sequel, we suppose that some gauge fixing mechanism will be implemented in order that the kernel of the kinetic term can be inverted.

ρ_2, i.e. the terms contained in the expressions S_{HH} and S_{HF} (2.30). The divergent function ρ_1^{-2} enters in both the Fourier transform of these expressions, which are given respectively in Appendix C by (C.1) and (C.2). In order to regularize them, we assume that the field ϕ is in fact a smeared field defined by

$$\phi = \int dz\ \rho_3(x,z)\varphi(z). \tag{2.32}$$

In addition, one requires, 1) that the function ρ_3, which enters in the definition (2.32) of the smeared field ϕ, behaves like

$$\rho_3(k) \sim \rho_1^2(k), \tag{2.33}$$

as the cutoff Λ goes to infinity, and 2) that the divergent cutoff function $\rho_1^{-1}(k)$ occurring in the expressions (C.1) and (C.2), must be multiplied by the larger number of the smeared field ϕ which is a function of the variable k. This means that, in order to calculate the Fourier transform of the expressions S_{HH} and S_{HF}, the Dirac delta function that reflects the momentum conservation at each vertex, must be integrated over the momentum variable of the ϕ field first. In that case, as explained in Appendix C, the shift of the momentum variable of the field ϕ is in general not allowed in some terms of the expressions (C.1) and (C.2), if these terms become not regularized.

Keeping in mind the rules outlined before, we see that both the expressions (C.1) and (C.2) are indeed regularized in the gauge field sector, if the propagator of the gluon field is a rapidly decreasing function in Euclidean momentum space. Moreover, given in Appendix C, the explicit form in momentum space of all the interaction terms between the gauge field A_μ and the smeared bosons fields ϕ (2.32), which contribute to the action (2.27) and (2.30), it is clear that the action (2.27) is now completely regularized in the gauge and scalar fields sector as long as the propagators of the φ fields (2.32) are also rapidly decreasing functions in Euclidean momentum space.

Up to now the action (2.27) does not contain any sector that is purely quadratic in the scalar field φ. In order to define the gluon and scalar field propagators, we will add to the action a new term, which is invariant under both the transformations (2.3) and (2.17) of the gauge and scalar fields. This invariant term enables us to invert the quadratic part of the gauge field A_μ. Since this new term contains a quadratic part in the field φ, which does not enforce the regularization, we will also add to the action (2.27) a mass term for the fields φ which will render the theory completely regularized.

3 The scalar field sector

Our aim is to add to the action (2.27) a new term in order that the gluon kinetic term can be inverted without explicitly breaking the invariance under both the gauge transformations

(2.3) and (2.17) . The new action, thus constructed, must be gauge-invariant in all the range of the cutoff Λ, regularized and must converge to the standard one of QCD as the cutoff goes to infinity. Because the part (2.27) of this new action is already converging to the standard form of QCD as the cutoff goes to infinity, the searched new term must vanish as the cutoff goes to infinity. Then, the part of the action which will give rise to the covariant form of the gauge field propagator can be defined as

$$S_{Gauge} = \frac{\xi}{g^2}\,\mathrm{Tr}\int dxdydz\,(\rho_1^{(-1)}(z,x)-\rho_2(z,x))(\rho_1^{(-1)}(z,y)-\rho_2(z,y))\partial^\mu \mathcal{A}_\mu^{Inv}\partial_y^\nu \mathcal{A}_\nu^{Inv}(y), \quad (3.1)$$

the gauge-invariant operator $\mathcal{A}_\mu^{Inv}$ being defined in (2.1), the scalar functions $\rho_1^{(-1)}$ and ρ_2 being respectively defined in (2.7) and (2.6), and ξ is a real parameter. If we recall that the operator U which enters in the definition of $\mathcal{A}_\mu^{Inv}$ is expressed in terms of the smeared field ϕ (2.32) by (2.13), we obtain in momentum space [6]

$$\begin{aligned} S_{Gauge} &= \frac{\xi}{g^2}\,\mathrm{Tr}\int d\bar{k}\,(\rho_1^{-1}(k)-\rho_2(k))^2\Big[k_\mu k_\nu A^\mu(k)A^\nu(-k)+\frac{k^4}{4\Lambda^2}(\phi^+(k)-\phi(-k)) \\ &\quad \times(\phi^+(-k)-\phi(k))\Big]+S_{Gauge}(A,\phi), \end{aligned} \quad (3.2)$$

where the term $S_{Gauge}(A,\phi)$ is explicitly given by the expression (C.3). Adding both the quadratic part in the gluons fields of the expression (3.2), to that of the regularized YM action S_{YM} (2.28), allows us to invert the gauge field kinetic term, and thus to define the free gluon propagator. This propagator is given in momentum space by

$$\langle A_{\mu\alpha}A_{\nu\beta}\rangle = -\rho_1^2(k)\frac{1}{k^4}\delta_{\alpha\beta}\big[(k^2 g_{\mu\nu}-k_\mu k_\nu)+\frac{1}{\xi(1-\rho_1(k)\rho_2(k))^2}k_\mu k_\nu\big]. \quad (3.3)$$

What happens in the pure scalar fields sector ? We define the smeared field ϕ in terms of the smeared fields σ and π by

$$\phi = \sigma + i\pi, \quad (3.4)$$

and, in the same way that the smeared field ϕ (2.14) is associated to the field φ in (2.32), we define the anti-Hermitic matrices ς and ϖ as

$$\varphi = \varsigma + i\varpi, \quad (3.5)$$

whose components ς_i and ϖ_i, on the basis formed by T_0 and the eight generators T_α, are real scalar fields. Because the fields ς_a and ϖ_a are real, we have,

$$\varsigma^+(k) = -\varsigma(-k) \quad (3.6)$$

$$\varpi^+(k) = -\varpi(-k). \quad (3.7)$$

[6] Remember that the limit $\eta \to 0$ must be taken in the Fourier transformed of the function $\rho_1^{(-1)}$.

In that case, the second term of (3.2) will give only a contribution to the propagator of the ς field. Up to now, since there is no other quadratic term in the pure scalar fields sectors of the parts of the action given in (2.27) and (3.2), and knowing that the cutoff function ρ_3 which enters in the definition of the smeared σ field (3.4) behaves like (2.33), we find that the free propagator of the ς field (3.5) will behave respectively in the IR and UV domains as

$$\langle \varsigma\varsigma \rangle \sim \tfrac{\Lambda^2}{k^4}\rho_1^{-2}(k). \tag{3.8}$$

Such a behavior for the ς field will then spoil the UV regularization of the theory and worse will enforce the IR divergences. Moreover a perturbative calculation cannot be performed because in the actions (2.27) and (3.2) there is no quadratic term in the ϖ fields (3.5). To cure this disease, we will add to the action a mass term for both ς and ϖ fields, while preserving if possible the invariance of the action under both the transformations (2.3) and (2.17).

We will first construct a mass invariant term for the field ϖ which is proportional to the sum $\phi^+ + \phi$. Since the operator

$$U^+U - I = \tfrac{1}{\Lambda}(\phi^+ + \phi) + \tfrac{1}{\Lambda^2}\phi^+\phi, \tag{3.9}$$

where U is defined by (2.13), is invariant under the transformation (2.17), the searched mass term will arise from the following action

$$S_\pi = -\tfrac{M^2}{4g^2}\,\mathrm{Tr}\int dxdy\ \rho_1^{(-6)}(x,y)(\phi^+ + \phi + \tfrac{1}{\Lambda}\phi^+\phi)(\phi^+(y) + \phi(y) + \tfrac{1}{\Lambda}\phi^+(y)\phi(y)), \tag{3.10}$$

where the function $\rho_1^{(-6)}$ is defined in (2.7) and M is a mass parameter. Therefore if we use the definitions (2.32) and (3.5), and the relations (3.6) and (3.7), and suppose for simplicity that the cutoff function ρ_3 (2.33), which enters in the definition of the smeared fields σ and π (3.4), is in fact given by

$$\rho_3(k) = \rho_1^2(k), \tag{3.11}$$

the action (3.10) becomes in momentum space

$$\begin{aligned} S_\pi \;=\; & \tfrac{M^2}{g^2}\,\mathrm{Tr}\Big\{ \int d\bar{k}\ \rho_1^{-2}(k)\varpi(k)\varpi(-k) - \tfrac{i}{\Lambda}\int d\bar{k}d\bar{p}\ \varphi^+(p)\varphi(p+k)\varpi(-k) \\ & -\tfrac{1}{4\Lambda^2}\int d\bar{k}d\bar{p}d\bar{q}\ \rho_1^2(k)\varphi^+(p)\varphi(p-k)\varphi^+(q)\varphi(q+k)\Big\}. \end{aligned} \tag{3.12}$$

In order that the second term of (3.12) gives rise only to regularized contributions, the free propagators of the ς and ϖ fields must be rapidly decreasing functions in Euclidean momentum space.

Regarding the ς field, there exists a priori no mass term invariant under the transformation (2.17). For simplicity, we choose the following term,

$$S_\sigma = \tfrac{M^2}{4g^2} \operatorname{Tr} \int dxdy \ \rho_1^{(-6)}(x,y)(\phi^+ - \phi)(\phi^+(y) - \phi(y)), \tag{3.13}$$

for the ς field mass term, which is given in momentum space by

$$S_\sigma = \tfrac{M^2}{g^2} \operatorname{Tr} \int d\bar{k} \ \rho_1^{-2}(k)\varsigma(k)\varsigma(-k). \tag{3.14}$$

We will see in the following, that the breaking induced by the term (3.14) does not really spoils the gauge invariance of the theory. Now from the relations (3.2) and (3.14), we obtain for the free propagator of the ς field

$$\langle \varsigma\varsigma \rangle = -\frac{1}{\xi \frac{k^4}{\Lambda^2}\rho_1^2(k)(1-\rho_1(k)\rho_2(k))^2 + M^2\rho_1^{-2}(k)}. \tag{3.15}$$

From the quadratic term of the action (3.12), and from the relation (3.15), we conclude that the free propagators of the ϖ and ς fields have the requested behavior in the IR and UV domain, i.e.

$$\langle \varpi\varpi \rangle \sim \langle \varsigma\varsigma \rangle \sim \tfrac{\rho_1^2(k)}{M^2}. \tag{3.16}$$

4 The regularized gauge-invariant path integral

From the preceding sections, we conclude that the form of the regularized action of the YM fields is given in adding to the action (2.27) both the contributions (3.1) of the gauge sector and that of the mass terms (3.10) and (3.13) of the ϖ and ς fields. Then the regularized action is

$$S_{Reg}(A, \varphi^+, \varphi, \Lambda) = S_{Inv}(A, \varphi^+, \varphi, \Lambda) + S_\sigma, \tag{4.1}$$

where

$$S_{Inv}(A, \varphi^+, \varphi, \Lambda) = S_{YM} + S_{A\phi^+\phi} + S_{Gauge} + S_\pi, \tag{4.2}$$

is invariant under the gauge transformations (2.3) and (2.4), and we recall that Λ is the cutoff scale and M is the mass of the ϖ and ς fields. In the configuration space the regularized path integral which is normalized to unity reads

$$Z_{Reg}(J, \Lambda) = \frac{\int \mathcal{D}A_\mu \mathcal{D}\varphi^+ \mathcal{D}\varphi \ e^{i\left(S_{Reg}(A,\varphi^+,\varphi,\Lambda) + S(J)\right)}}{\int \mathcal{D}A_\mu \mathcal{D}\varphi^+ \mathcal{D}\varphi \ e^{iS_{Reg}(A,\varphi^+,\varphi,\Lambda)}}, \tag{4.3}$$

where $S(J)$ is given as usual in terms of the external sources J_μ for the gluons fields by

$$S(J) = -\tfrac{2}{g} \operatorname{Tr} \int dx \ J_\mu A^\mu. \tag{4.4}$$

By construction, when the cutoff Λ goes to infinity, the action (4.1) has the property

$$\lim_{\Lambda\to\infty} S_{Reg}(A,\varphi^+,\varphi,\Lambda) = \tfrac{1}{2g^2}\,\mathrm{Tr}\int dx F^{\mu\nu}F_{\mu\nu} - \tfrac{M^2}{g^2}\,\mathrm{Tr}\int dx \varphi^+\varphi. \tag{4.5}$$

This tells us that the auxiliary bosons fields φ will decouple from the theory. We will show that this decoupling occurs when the mass M of the bosons fields φ are at least greater or equal to the cutoff scale Λ.

After a loopwise expansion of the path integral (4.3) with respect to the part of the action which contains the vertices of the φ fields and integration over these fields, as show in Appendix D, the proper vertex Γ (D.7) associated to any gluons effective amplitude behaves like

$$\Gamma \sim \left(\tfrac{\Lambda^2}{M^2}\right)^{I_\varphi - n}\mathcal{O}(\Lambda^{-k}), \tag{4.6}$$

as all the internal momenta scale with the cutoff Λ. Here n and I_φ are respectively the number of pure φ fields vertices and the number of internal φ fields occurring in Γ. As show in Appendix D, the number k (D.8) is a positive integer for all the gluons effective vertices, except for the two, three and four legs effective vertices, which are proportional to the vertices of the pure Yang-Mills action (C.4), where k is zero. As a result if

$$1 \ll \Lambda^2 \leq M^2 \quad \text{or} \quad 0 \leq \Lambda^2 \ll M^2, \tag{4.7}$$

the auxiliary fields φ decouple from the gauge sector. Notice that the rescaling

$$\phi \;\rightarrow\; \Lambda\phi, \tag{4.8}$$

of the smeared field ϕ (2.32) does not change this conclusion. This is because, in each internal φ boson line the rescaling of the field is exactly compensated by the rescaling

$$M \;\rightarrow\; \Lambda M, \tag{4.9}$$

of the mass M entering in the propagator, so that the behavior of the proper effective vertex Γ (4.6) is left unchanged. In order to have only one scale in the regularized path integral (4.3), a good choice is to take for the mass M of the φ fields the cutoff scale Λ.

If we denote ρ_1 by ρ, and remember that the cutoff function ρ_2 is constrained by the relation (2.31), and that we have fixed the function ρ_3 by the equation (3.11), the action (4.1) is still regularized if we choose the following relations between ρ_2 and ρ_3

$$\rho_2(k) = \rho_3(k) = \rho(k)^2. \tag{4.10}$$

What are the WT identities that can be deduced from the expression of the path integral (4.3) ? The WT identities follow from the invariance of the path integral (4.3) under the

shift of variable induces by the infinitesimal form of the gauge transformation (2.3) and the smeared field transformation (2.17), i.e.

$$\begin{aligned} A^\mu_\alpha &\rightarrow A^\mu_\alpha - C_{\alpha\beta\gamma}A^\mu_\beta\epsilon_\gamma + \partial_\mu\epsilon_\alpha && (4.11)\\ \phi &\rightarrow \phi + \epsilon\phi + \Lambda\epsilon, && (4.12)\end{aligned}$$

where we use the shorthand notation $\epsilon \equiv \epsilon_\alpha T^\alpha$. By construction, all the terms of the action (4.1), except the term S_σ, are invariant under both the gauge transformation (2.3) and the smeared field transformation (2.17). In addition the measure $\mathcal{D}\varphi^+\mathcal{D}\varphi$ is also invariant under the transformations of the fields φ which are induced by the transformations (2.17) of the smeared field ϕ (2.32). This is because, 1) the measure $\mathcal{D}\varphi^+\mathcal{D}\varphi$ is proportional[7] to the measure $\mathcal{D}\phi^+\mathcal{D}\phi$ and 2) the transformation (2.17) of the smeared field ϕ (2.32) leaves the measure (2.22) invariant.

In order to derive the WT identities for the gluons fields, we will show that the symmetry breaking term S_σ (3.13) is harmless. In fact, if we introduce the auxiliary matrix valued field

$$\varrho = \varrho_a T^a, \tag{4.13}$$

whose components ϱ_a are real scalar fields, and define the action

$$S_{\sigma\varrho} = -\frac{M^2}{g^2}\,\mathrm{Tr}\int dy\big[2\varrho(y)\int dx\rho^{(-1)}(x,y)\sigma + \varrho^2(y)\big], \tag{4.14}$$

the path integral (4.3) can be written as

$$Z_{Reg}(J,\Lambda) = \frac{\int \mathcal{D}A_\mu\mathcal{D}\varphi^+\mathcal{D}\varphi\mathcal{D}\varrho\ e^{i\left(S_{Inv}(A,\varphi^+,\varphi,\Lambda)+S_{\sigma\varrho}+S(J)\right)}}{\int \mathcal{D}A_\mu\mathcal{D}\varphi^+\mathcal{D}\varphi\mathcal{D}\varrho\ e^{i(S_{Inv}(A,\varphi^+,\varphi,\Lambda)+S_{\sigma\varrho})}}. \tag{4.15}$$

This is because the Gaussian integration over the ϱ fields transforms the action $S_{\sigma\varrho}$ (4.14) in the action S_σ (3.14). Since under the infinitesimal transformation (4.12) the components of the smeared fields σ and π are mixed in the following way

$$\begin{aligned} \delta\sigma_0 &= -\tfrac{1}{\sqrt{6}}\pi_\alpha\epsilon_\alpha && (4.16)\\ \delta\sigma_\alpha &= -\tfrac{1}{2}(C_{\alpha\beta\gamma}\sigma_\beta + d_{\alpha\beta\gamma}\pi_\beta)\epsilon_\gamma - \tfrac{1}{\sqrt{6}}\pi_0\epsilon_\alpha + \Lambda\epsilon_\alpha \\ \delta\pi_0 &= \tfrac{1}{\sqrt{6}}\sigma_\alpha\epsilon_\alpha && (4.17)\\ \delta\pi_\alpha &= -\tfrac{1}{2}(C_{\alpha\beta\gamma}\pi_\beta - d_{\alpha\beta\gamma}\sigma_\beta)\epsilon_\gamma + \tfrac{1}{\sqrt{6}}\sigma_0\epsilon_\alpha,\end{aligned}$$

the infinitesimal variation of the action $S_{\sigma\varrho}$ (4.14) under the transformation (4.12) is given by

$$\begin{aligned}\delta S_{\sigma\varrho} &= -\frac{M^2}{g^2}\int dxdy\rho^{(-1)}(x,y)\Big[\tfrac{1}{\sqrt{6}}\varrho_0(y)\epsilon_\alpha\pi_\alpha + \varrho_\beta(y)\big\{\tfrac{1}{2}\epsilon_\alpha(\sigma_\gamma C_{\alpha\beta\gamma} + \pi_\gamma d_{\alpha\beta\gamma}) \\ &\quad + (\tfrac{1}{\sqrt{6}}\pi_0 - \Lambda)\epsilon_\beta\big\}\Big]. && (4.18)\end{aligned}$$

[7]This fact is readily seen in momentum space.

The invariance under the infinitesimal transformations (4.11), (4.16) and (4.17) of the path integral (4.15) gives the following identity

$$\langle \tfrac{\delta S_{\sigma\varrho}}{\delta\epsilon_\alpha} - \tfrac{1}{g}(C_{\alpha\beta\gamma}J^\mu_\beta A_{\mu\gamma} + \partial_\mu J^\mu_\alpha)\rangle = 0. \tag{4.19}$$

In a intermediate step, and in order to express the connected Green's functions, we introduce respectively in the path integral (4.15) the external sources K_a, L_a and N_a for the scalar fields ς_a, ϖ_a (3.5) and ϱ_a (4.13), in replacing the source term $S(J)$ (4.4) by the expression

$$S(J,K,L,N) = -\tfrac{2}{g}\,\mathrm{Tr}\int dx\ (J_\mu A^\mu + K\varsigma + L\varpi + N\varrho). \tag{4.20}$$

Then, the generating functional of connected Green's functions in the presence of the external sources for the gauge and scalar fields is defined as

$$Z_{Reg}(J,K,L,N,\Lambda) = e^{W_{Reg}(J,K,L,N,\Lambda)}. \tag{4.21}$$

As usual, we define for each field its vacuum expectation value in the presence of the external source. These C-number functions are obtained by taking the functional derivative of W_{Reg} with respect to their own sources. We also recall that the generating functional of 1PI functions is defined in terms of the vacuum expectation values of the fields by the Legendre transformation

$$\Gamma_{Reg}(A,\varsigma,\varpi,\varrho,\Lambda) = -iW_{Reg}(J,K,L,N,\Lambda) + \tfrac{2}{g}\,\mathrm{Tr}\int dx\ (J_\mu A^\mu + K\varsigma + L\varpi + N\varrho). \tag{4.22}$$

In that case there is a one-to-one correspondence between the classical field thus obtained and its external source. For instance these conjugate relations hold true for the fields A_μ and ϱ

$$A^\mu_\alpha = -ig\frac{\delta W_{Reg}}{\delta J_{\mu\alpha}} \qquad J_{\mu\alpha} = -\,g\frac{\delta\Gamma_{Reg}}{\delta A^\mu_\alpha} \tag{4.23}$$

$$\varrho_a = -ig\frac{\delta W_{Reg}}{\delta N_a} \qquad N_a = -\,g\frac{\delta\Gamma_{Reg}}{\delta\varrho_a} \tag{4.24}$$

We are now able to show that the functional derivative of $\langle\frac{\delta S_{\sigma\varrho}}{\delta\epsilon_\alpha}\rangle$ (4.18) with respect to the vacuum expectation value of the gauge field A_μ vanishes identically.

At first, when taking the functional derivative of $\langle\frac{\delta S_{\sigma\varrho}}{\delta\epsilon_\alpha}\rangle$ (4.18) with respect to the classical field A_μ, the terms which are only proportional to $\langle\varrho\rangle$ do not contribute. This is because, 1) the vacuum expectation values of the fields A_μ and ϱ are by construction independent variables, and 2) at the end of the calculation we will set to zero the external sources relative to the scalar fields, Then, using the chain rule which expresses the functional derivative $\frac{\delta}{\delta A}$ in term of $\frac{\delta}{\delta J}$, it remains to show that the functional derivative of the connected Green's functions $\langle\varrho_a\pi_b\rangle$ and $\langle\varrho_a\sigma_b\rangle$ with respect to the gauge fields sources J_μ vanish effectively.

The equation of motion of the auxiliary field ϱ, which is deduced from the expressions of the generating functional (4.21) and of the action $S_{\sigma\varrho}$ (4.14), can be expressed with the help of the relations (4.24) in terms of the classical fields as

$$M^2[\int dy\ \rho(x,y)\varsigma_a(y) + \varrho_a] - g^2\frac{\delta\Gamma_{Reg}}{\delta\varrho_a} = 0. \tag{4.25}$$

If we take the functional derivative of both sides of the equation (4.25), first with respect to the classical fields $\varpi_b(y)$ or $\varsigma_b(y)$, and then with respect to $A^\mu_\alpha(z)$, we get

$$\begin{aligned} \frac{\delta^3\Gamma_{Reg}}{\delta A^\mu_\alpha(z)\delta\varpi_b(y)\delta\varrho_a} &= 0 \\ \frac{\delta^3\Gamma_{Reg}}{\delta A^\mu_\alpha(z)\delta\varsigma_b(y)\delta\varrho_a} &= 0. \end{aligned} \tag{4.26}$$

Because π and σ are respectively the smeared field associated to the scalar field ϖ and ς, the vanishing of the 1PI functions (4.26) shows that the functional derivatives of the connected Green's functions $\langle\varrho_a\pi_b\rangle$ and $\langle\varrho_a\sigma_b\rangle$ with respect to the source J of the gauge field also vanish identically, whatever the values of the scalar fields sources. As a result the functional derivative of the vacuum expectation value $\langle\frac{\delta S_{\sigma\varrho}}{\delta\epsilon_\alpha}\rangle$ (4.18) with respect to the classical field A_μ, indeed vanishes. Then, in expressing the WT identity (4.19) in terms of 1PI functions, we get the following relation

$$\frac{\delta^2\Gamma_{Reg}}{\delta A^\nu_\beta(y)\delta A^\mu_\gamma}C_{\alpha\gamma\delta}A^\mu_\delta - C_{\alpha\beta\gamma}\frac{\delta\Gamma_{Reg}}{\delta A^\nu_\gamma}\delta(x-y) + \partial^\mu\frac{\delta^2\Gamma_{Reg}}{\delta A^\mu_\alpha\delta A^\nu_\beta(y)} = 0. \tag{4.27}$$

In this identity Γ_{Reg} is the generating functional of 1PI functions defined by

$$\Gamma_{Reg}(A,\Lambda) = -iW_{Reg}(J,\Lambda) + \frac{2}{g}\operatorname{Tr}\int dx\ J_\mu A^\mu, \tag{4.28}$$

which means that the integration over the scalar fields φ has already been done, at least implicitly, in the path integral (4.3). The regularized WT (4.27) is our main result, and shows that all regularized 1PI functions which are deduced from the regularized path integral (4.3) are all transverse with respect to the gauge field when the external source is switched off.

5 Conclusion and outlook

We have shown that the path integral of pure YM theory can be regularized in four-dimensional physical space in a gauge-invariant manner. The regularization is implemented nonperturbatively at the level of the action through cutoff functions. The price to pay, in order to maintain the original gauge invariance, is to enlarge the field content of the theory by a set of auxiliary complex scalar fields that play the role of Stueckelberg compensating fields. By contrast with any PV inspired regularization, the Gaussian metric induced by

these fields has the good sign, and the path integral is here a mathematically well-defined object. Since the action, and therefore the path integral of the theory, is regularized nonperturbatively in all the energy range, this regularization will be useful to study the evolution of physical quantities with the energy scale, without making any assumptions for matching of the quantities calculated in the IR and in the UV domains, as is the case in lattice gauge theory. In a forthcoming paper, we shall study first the consequences of this regularization in perturbation theory, and then beyond.

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Appendix A

Here for convenience and to fix our conventions we recall the commutation relations of the generators (2.2) of the group $SU(3)$

$$\begin{aligned} [T^\alpha, T^\beta] &= C_{\alpha\beta\gamma} T^\gamma && \text{(A.1)} \\ \{T^\alpha, T^\beta\} &= -\tfrac{1}{3}\delta_{\alpha\beta} + i d_{\alpha\beta\gamma} T^\gamma, && \text{(A.2)} \end{aligned}$$

the structures constants $C_{\alpha\beta\gamma}$ having the opposite sign of those defined in [18]. We recall that the trace properties of these matrices follow recursively from the relation

$$T^\alpha T^\beta = -\tfrac{1}{6}\delta_{\alpha\beta} + \tfrac{1}{2}(C_{\alpha\beta\gamma} + i d_{\alpha\beta\gamma}) T^\gamma. \tag{A.3}$$

For instance the trace of the product of three and four generators are given by

$$\begin{aligned} \mathrm{Tr}(T^\alpha T^\beta T^\gamma) &= -\tfrac{1}{4}(C_{\alpha\beta\gamma} + i d_{\alpha\beta\gamma}) && \text{(A.4)} \\ \mathrm{Tr}(T^\alpha T^\beta T^\gamma T^\delta) &= \tfrac{1}{12}\delta_{\alpha\beta}\delta_{\gamma\delta} - \tfrac{1}{8}(C_{\alpha\beta\alpha'} + i d_{\alpha\beta\alpha'})(C_{\gamma\delta\alpha'} + i d_{\gamma\delta\alpha'}). && \text{(A.5)} \end{aligned}$$

Appendix B

To have an explicit expression of the transformation of the scalar fields ϕ_i induced by the matrix transformation (2.17), we recast this equality between two 3×3 complex matrices as an equality between two complex vectors having nine components. Let C_V be the nine components vector column obtained from any 3×3 matrix V by putting the successive columns of the matrix V as a single column in increasing order from top to bottom. Then, the tensorial product involving the complex 3×3 matrix V

$$T(V) = I \otimes V, \tag{B.1}$$

is a 9×9 matrix $T(V)$ that relates the constant column vector C_I associated with the 3×3 identity matrix I to the column vector C_V as,

$$C_V = T(V)C_I. \tag{B.2}$$

By construction, the linear mapping T between the set of 3×3 complex matrices V and the set of 9×9 complex matrices $T(V)$, is an group isomorphism, that is to say, for any matrices V and V' the relations

$$T(VV') = T(V)T(V'), \quad T(I) = I \quad \text{and} \quad T(V^+) = T^+(V) \tag{B.3}$$

hold. Moreover, due to the definition (B.1) we have the obvious relation

$$\det T(V) = (\det V)^3. \tag{B.4}$$

Using the definition (B.2) and the properties (B.3), the vectorial form of the equation (2.17) is then given by the equation

$$T(\phi')C_I = T(V)T(\phi)C_I + \Lambda T(V - I)C_I. \tag{B.5}$$

We can write the vector $T(\phi)C_I$ associated to the matrix (2.14) as

$$T(\phi)C_I = S\Phi, \tag{B.6}$$

where S is an invertible 9×9 constant matrix acting on the vector column Φ, whose components are the fields ϕ_i defined in (2.14), the index i running from 0 to 8. In that case, we can rewrite the equation (B.5) as

$$S\Phi' = T(V)S\Phi + \Lambda T(V - I)C_I. \tag{B.7}$$

Appendix C

In this appendix we give explicitly the terms of the action (2.27) in momentum space. In all the terms, the integration over the Dirac's delta functions which result from the translational invariance of the action are first performed with respect to the momenta of the ϕ field, and owing to the conditions (2.31) and (2.33) the product of cutoff functions $\rho_1^{-1}\rho_2$ and $\rho_1^{-1}\rho_3$ are rapidly decreasing functions of the square of the momenta in Euclidean space. Moreover, as said before, the inverse Fourier transform of the scalar function (2.7) $\rho_1^{(-1)}$ is defined when the infinitesimal parameter η tends to zero. After a lengthy but straightforward calculation,

we obtain respectively for the terms S_{HH}, S_{HF} and S_{Gauge} which are defined in (2.29) , (2.30) and (3.2).

$$\begin{aligned}S_{HH} &= \tfrac{1}{g^2}\,\mathrm{Tr}\int d\bar{k}d\bar{p}d\bar{q}\ (\rho_1^{-1}(k)-\rho_2(k))^2\Big\{(k^2q_\mu - kqk_\mu)A^\mu_\alpha(p)\Big[\tfrac{2i}{\Lambda^3}(\phi^+(p-k)T^\alpha \\ &+T^\alpha\phi(k-p))\phi^+(q)\phi(q-k)+\tfrac{2i}{\Lambda^4}\int d\bar{p}'\phi^+(p')T^\alpha\phi(p'+k-p)\phi^+(q)\phi(q-k)\Big] \\ &+(k^2g_{\mu\nu}-k_\mu k_\nu)A^\mu_\alpha(p)A^\nu_\beta(q)\Big[\tfrac{1}{\Lambda^2}(\phi^+(p-k)T^\alpha+T^\alpha\phi(k-p))(\phi^+(q+k)T^\beta \\ &+T^\beta\phi(-q-k))+\tfrac{2}{\Lambda^3}\int d\bar{p}'(\phi^+(p-k)T^\alpha+T^\alpha\phi(k-p))\phi^+(p')T^\beta\phi(p'-k-q) \\ &+\tfrac{1}{\Lambda^4}\int d\bar{p}'d\bar{q}'\phi^+(p')T^\alpha\phi(p'+k-p)\phi^+(q')T^\beta\phi(q'-k-q)\Big] \\ &-\tfrac{1}{\Lambda^4}p^\mu(k^2q_\mu-kqk_\mu)\phi^+(p)\phi(k+p)\phi^+(q)\phi(q-k)\Big\}\end{aligned} \tag{C.1}$$

$$\begin{aligned}S_{HF} &= \tfrac{2}{g^2}\,\mathrm{Tr}\int d\bar{p}d\bar{q}\ \rho_1^{-1}(p)(\rho_1^{-1}(p)-\rho_2(p))\Big\{\tfrac{i}{\Lambda^2}(p^2q_\mu-pqp_\mu)A^\mu_\alpha(p)T^\alpha\phi^+(q)\phi(q-p) \\ &+(p^2g_{\mu\nu}-p_\mu p_\nu)A^\mu_\alpha(p)A^\nu_\beta(q)\Big[\tfrac{1}{\Lambda}(T^\alpha T^\beta\phi(-p-q)+T^\beta T^\alpha\phi^+(p+q)) \\ &+\tfrac{1}{\Lambda^2}\int d\bar{p}'T^\alpha\phi^+(p')T^\beta\phi(p'-p-q)\Big]\Big\}-\tfrac{2}{g^2}\,\mathrm{Tr}\int d\bar{p}d\bar{q}d\bar{r}\rho_2(q+r)(\rho_1^{-1}(q+r) \\ &-\rho_2(q+r))\Big\{\tfrac{1}{\Lambda^2}p_\nu(q+r)_\mu A^\nu_\beta(q)A^\mu_\gamma(r)\phi^+(p)\phi(p-q-r)[T^\beta,T^\gamma] \\ &+(q+r)_\nu A^\mu_\alpha(p)A^\nu_\beta(q)A_{\gamma\mu}(r)\Big[\tfrac{i}{\Lambda}(\phi^+(p+q+r)T^\alpha+T^\alpha\phi(-p-q-r)) \\ &+\tfrac{i}{\Lambda^2}\int d\bar{p}'\phi^+(p')T^\alpha\phi(p'-p-q-r)\Big][T^\beta,T^\gamma]\Big\}\end{aligned} \tag{C.2}$$

$$
\begin{aligned}
S_{Gauge}(A,\phi) \quad = \quad & \tfrac{\xi}{g^2}\,\mathrm{Tr}\int d\bar{k}\;(\rho_1^{-1}(k)-\rho_2(k))^2\Big\{\int d\bar{p}k_\mu A^\mu_\alpha(p)\Big[\tfrac{i}{\Lambda^2}k^2(\phi^+(p-k)T^\alpha \\
& +T^\alpha\phi(k-p))(\phi^+(k)-\phi(-k))+\tfrac{i}{\Lambda^3}\int d\bar{q}k^2\phi^+(q)T^\alpha\phi(q+k-p) \\
& \times(\phi^+(k)-\phi(-k))+\tfrac{i}{\Lambda^3}\int d\bar{q}(q^2-(k-q)^2)(\phi^+(p-k)T^\alpha \\
& +T^\alpha\phi(k-p))\phi^+(q)\phi(q-k)+\tfrac{i}{\Lambda^4}\int d\bar{p}'d\bar{q}(q^2-(k-q)^2)\phi^+(p')T^\alpha \\
& \times\phi(p'+k-p)\phi^+(q)\phi(q-k)\Big]+\int d\bar{p}d\bar{q}k_\mu k_\nu A^\mu_\alpha(p)A^\nu_\beta(q)\Big[\tfrac{1}{\Lambda^2}(\phi^+(p-k)T^\alpha \\
& +T^\alpha\phi(k-p))(\phi^+(k+q)T^\beta+T^\beta\phi(-k-q))+\tfrac{2}{\Lambda^3}\int d\bar{p}'\;(\phi^+(p-k)T^\alpha \\
& +T^\alpha\phi(k-p))\phi^+(p')T^\beta\phi(p'-k-q)+\tfrac{1}{\Lambda^4}\int d\bar{p}'d\bar{q}'\;\phi^+(q')T^\alpha\phi(k-p+q') \\
& \times\phi^+(p')T^\beta\phi(p'-k-q)\Big]+\int d\bar{p}(p^2-(k+p)^2)\Big[\tfrac{1}{2\Lambda^3}k^2\phi^+(p)\phi(k+p) \\
& \times(\phi^+(k)-\phi(-k))+\tfrac{1}{4\Lambda^4}\int d\bar{q}(q^2-(q-k)^2)\phi^+(p)\phi(k+p)\phi^+(q)\phi(q-k)\Big] \\
& +k_\mu A^\mu_\alpha(k)\Big[\tfrac{i}{\Lambda}k^2T^\alpha(\phi^+(k)-\phi(-k))+\tfrac{i}{\Lambda^2}\int d\bar{p}((k+p)^2-p^2)T^\alpha\phi^+(k+p) \\
& \times\phi(p)\Big]+\int d\bar{p}k_\mu k_\nu A^\mu_\alpha(k)A^\nu_\beta(p)\Big[\tfrac{2}{\Lambda}T^\alpha(\phi^+(k+p)T^\beta+T^\beta\phi(-k-p)) \\
& +\tfrac{2}{\Lambda^2}\int d\bar{q}T^\alpha\phi^+(q)T^\beta\phi(q-k-p)\Big]\Big\} \qquad\qquad (\mathrm{C.3})
\end{aligned}
$$

The divergent cutoff function ρ_1^{-2} enters in both the expressions (C.1) , (C.2) and (C.3). The shift of the momentum variable of the field ϕ is in general not allowed in some terms of the expressions (C.1), (C.2) or (C.3), if these terms become not regularized. For example, let us consider the expression (C.1).

If we perform the shift of variables $p \to p+k$ and $q \to q-k$ in the term proportional to $\frac{1}{\Lambda^2}A^\mu_\alpha(p)A^\nu_\beta(q)$, its UV behavior becomes worse. This is because, the divergent function $\rho_1^{-2}(k)$, is now only multiplied by the product of the fields $A^\mu_\alpha(p+k)A^\nu_\beta(q-k)$, and not by any smeared field ϕ function of the variable k which adds the multiplicative factor $\rho_1^2(k)$. Having in mind the rules outlined before, one can convince oneself that the expressions (C.1), (C.2) and (C.3) are indeed regularized if the propagators of the gluons and of the φ fields (2.32) are rapidly decreasing functions in momentum space. Notice that the contributions to the action (2.27) of the terms in the second brace of (C.2) can be regularized without any constraints on the gluons and φ propagators, knowing that the product of cutoff functions $\rho_2(\rho_1^{-1}-\rho_2)$ is a rapidly decreasing function of k^2 in momentum Euclidean space.

The contribution to the action (2.27) of its pure YM part (2.28) and of the first, third and

last terms of (2.30), are respectively given in momentum space by the following expressions,

$$\begin{aligned} S_{YM} &= -\tfrac{1}{2g^2}\int d\bar{k}\ \rho_1^{-2}(k)(k^2 g_{\mu\nu} - k_\mu k_\nu)A^\mu_\alpha(k)A^\nu_\alpha(-k) \\ &+\tfrac{i}{g^2}C_{\alpha\beta\gamma}\int d\bar{k}d\bar{p}\ \rho_1^{-1}(k)\rho_2(k)k^\nu A^\mu_\alpha(k)A_{\mu\beta}(p)A_{\nu\gamma}(-k-p) \\ &-\tfrac{1}{4g^2}C_{\alpha\beta\alpha'}C_{\gamma\delta\alpha'}\int d\bar{k}d\bar{p}d\bar{q}\ \rho_2^2(k)A^\mu_\alpha(p)A^\nu_\beta(k-p)A_{\mu\gamma}(q)A_{\nu\delta}(-k-q) \end{aligned} \tag{C.4}$$

$$\begin{aligned} S_{GG} &= \tfrac{1}{g^2}\,\mathrm{Tr}\int d\bar{k}d\bar{p}d\bar{q}\ \rho_2^2(k)\Big\{-(pqg_{\mu\nu} - p_\nu q_\mu)A^\mu_\alpha(p)A^\nu_\beta(q)\Big[\tfrac{1}{\Lambda^2}(\phi^+(p-k)T^\alpha \\ &+T^\alpha\phi(k-p))(\phi^+(k+q)T^\beta + T^\beta\phi(-k-q)) + \tfrac{2}{\Lambda^3}\int d\bar{p}'(\phi^+(p-k)T^\alpha \\ &+T^\alpha\phi(k-p))\phi^+(p')T^\beta\phi(p'-k-q) + \tfrac{1}{\Lambda^4}\int d\bar{p}'d\bar{q}'\phi^+(p')T^\alpha\phi(p'+k-p) \\ &\times\phi^+(q')T^\beta\phi(q'-k-q)\Big] + \int d\bar{r}p_\nu A^\mu_\alpha(p)A^\nu_\beta(q)A_{\mu\gamma}(r)\Big[\tfrac{2i}{\Lambda^2}(\phi^+(p-k)T^\alpha \\ &+T^\alpha\phi(k-p))(\phi^+(k+q+r)[T^\beta,T^\gamma] + [T^\beta,T^\gamma]\phi(-k-q-r)) \\ &+\tfrac{2i}{\Lambda^3}\int d\bar{p}'(\phi^+(p-k)T^\alpha + T^\alpha\phi(k-p))\phi^+(p')[T^\beta,T^\gamma]\phi(p'-k-q-r) \\ &+\tfrac{2i}{\Lambda^3}\int d\bar{p}'\phi^+(p')T^\alpha\phi(p'-p-k)(\phi^+(q-k+r)[T^\beta,T^\gamma] + [T^\beta,T^\gamma] \\ &\times\phi(k-q-r)) + \tfrac{2i}{\Lambda^4}\int d\bar{p}'d\bar{q}'\phi^+(p')T^\alpha\phi(k+p'-p)\phi^+(q') \\ &\times[T^\beta,T^\gamma]\phi(q'-k-q-r)\Big] + \int d\bar{r}d\bar{s}A^\mu_\alpha(p)A^\nu_\beta(q)A_{\nu\gamma}(r)A_{\mu\delta}(s) \\ &\times\Big[\tfrac{1}{\Lambda^2}(\phi^+(p+q-k)T^\alpha T^\beta + T^\alpha T^\beta\phi(k-p-q))(\phi^+(k+r+s)[T^\delta,T^\gamma] \\ &+[T^\delta,T^\gamma]\phi(-k-r-s)) + \tfrac{2}{\Lambda^3}\int d\bar{p}'(\phi^+(p+q-k)T^\alpha T^\beta \\ &+T^\alpha T^\beta\phi(k-p-q))\phi^+(p')[T^\delta,T^\gamma]\phi(p'-k-r-s) + \tfrac{1}{\Lambda^4}\int d\bar{p}'d\bar{q}'\phi^+(p') \\ &\times T^\alpha T^\beta\phi(k+p'-p-q)\phi^+(q')[T^\delta,T^\gamma]\phi^+(q'-k-r-s)\Big]\Big\} \end{aligned} \tag{C.5}$$

$$
\begin{aligned}
S_{GF} &= \tfrac{2}{g^2}\,\mathrm{Tr}\int d\bar{k}d\bar{p}\ \rho_1^{-1}(k)\rho_2(k)\Big\{-(kpg_{\mu\nu}-k_\mu p_\nu)A^\mu_\alpha(p)A^\nu_\beta(k)\Big[\tfrac{1}{\Lambda}(\phi^+(k+p)T^\alpha \\
&\quad +T^\alpha\phi(-k-p))T^\beta+\tfrac{1}{\Lambda^2}\int d\bar{q}\phi^+(k+p+q)T^\alpha\phi(q)T^\beta\Big]+\int d\bar{q}k_\mu A^\mu_\alpha(p)A^\nu_\beta(q) \\
&\quad A_{\nu\gamma}(k)\Big[\tfrac{i}{\Lambda}(\phi^+(p+q+k)[T^\alpha,T^\beta]+[T^\alpha,T^\beta]\phi(-k-p-q))T^\gamma \\
&\quad +\tfrac{i}{\Lambda^2}\int d\bar{p}'\phi^+(p')[T^\alpha,T^\beta]\phi(p'-k-p-q)T^\gamma\Big]\Big\}+\tfrac{2}{g^2}\,\mathrm{Tr}\int d\bar{p}d\bar{q}d\bar{r}\rho_2^2(q+r) \\
&\quad \Big\{p_\nu A^\mu_\alpha(p)A^\nu_\beta(q)A_{\mu\gamma}(r)\Big[\tfrac{i}{\Lambda}(\phi^+(p+q+r)T^\alpha+T^\alpha\phi(-p-q-r)) \\
&\quad +\tfrac{i}{\Lambda^2}\int d\bar{p}'\phi^+(p')T^\alpha\phi(p'-p-q-r)\Big][T^\beta,T^\gamma]+\int d\bar{s}A^\mu_\alpha(p)A^\nu_\beta(q)A_{\mu\gamma}(r) \\
&\quad A^\nu_\delta(s)\Big[\tfrac{1}{\Lambda}\phi^+(p+q+r+s)T^\alpha T^\delta+T^\alpha T^\delta\phi(-p-q-r-s)) \\
&\quad -\tfrac{1}{\Lambda^2}\int d\bar{p}'\phi^+(p')T^\alpha T^\delta\phi(p'-p-q-r-s)\Big][T^\beta,T^\gamma]\Big\}
\end{aligned}
\tag{C.6}
$$

$$
\begin{aligned}
S_{GH} &= \tfrac{2}{g^2}\,\mathrm{Tr}\int d\bar{k}d\bar{p}d\bar{q}\ \rho_2(k)(\rho_1^{-1}(k)-\rho_2(k))\Big\{A^\mu_\alpha(p)\Big[\tfrac{i}{\Lambda^3}(kpq_\mu-qpk_\mu)(\phi^+(p-k)T^\alpha \\
&\quad +T^\alpha\phi(k-p))\phi^+(q)\phi(q-k)+\tfrac{i}{\Lambda^4}\int d\bar{p}'(kpp'_\mu-pp'k_\mu)\phi^+(q)T^\alpha\phi(k+q-p) \\
&\quad \times\phi^+(p')\phi(p'-k)\Big]+(kpg_{\mu\nu}-k_\mu p_\nu)A^\mu_\alpha(p)A^\nu_\beta(q)\Big[\tfrac{1}{\Lambda^2}(\phi^+(p-k)T^\alpha+T^\alpha\phi(k-p)) \\
&\quad \times(\phi^+(k+q)T^\beta+T^\beta\phi(-k-q))+\tfrac{1}{\Lambda^3}\int d\bar{p}'(\phi^+(p-k)T^\alpha+T^\alpha\phi(k-p)) \\
&\quad \times\phi^+(p')T^\beta\phi(p'-k-q)+\tfrac{1}{\Lambda^3}\int d\bar{p}'\phi^+(p')T^\alpha\phi(k+p'-p)(\phi^+(k+q)T^\beta \\
&\quad +T^\beta\phi(-k-q))+\tfrac{1}{\Lambda^4}\int d\bar{p}'d\bar{q}'\phi^+(p')T^\alpha\phi(k+p'-p)\phi^+(q')T^\beta\phi(q'-k-q)\Big] \\
&\quad +A^\mu_\alpha(p)A^\nu_\beta(q)\int d\bar{p}'(k_\mu p'_\nu-k_\nu p'_\mu)\Big[\tfrac{1}{\Lambda^3}(\phi^+(p+q-k)T^\alpha T^\beta+T^\alpha T^\beta\phi(k-p-q)) \\
&\quad \times\phi^+(p')\phi(p'-k)+\tfrac{1}{\Lambda^4}\int d\bar{q}'\phi^+(q')T^\alpha T^\beta\phi(k+q'-p-q)\phi^+(p')\phi(p'-k)\Big] \\
&\quad -i\int d\bar{r}k_\mu A^\mu_\alpha(p)A^\nu_\beta(q)A_{\nu\gamma}(r)\Big[\tfrac{1}{\Lambda^2}(\phi^+(p+q-k)[T^\alpha,T^\beta]+[T^\alpha,T^\beta]\phi(k-p-q)) \\
&\quad \times(\phi^+(k+r)T^\gamma+T^\gamma\phi(-k-r))+\tfrac{1}{\Lambda^3}\int d\bar{p}'(\phi^+(p+q-k)[T^\alpha,T^\beta]+[T^\alpha,T^\beta] \\
&\quad \times\phi(k-p-q))\phi^+(p')T^\gamma\phi(p'-r-k)+\tfrac{1}{\Lambda^3}\int d\bar{p}'\phi^+(p')[T^\alpha,T^\beta]\phi(k-p+p'-q) \\
&\quad \times(\phi^+(k+r)T^\gamma+T^\gamma\phi(-k-r))+\tfrac{1}{\Lambda^4}\int d\bar{p}'d\bar{q}'\phi^+(p')[T^\alpha,T^\beta]\phi(k-p+p'-q) \\
&\quad \times\phi^+(q')T^\gamma\phi(q'-k-r)\Big]\Big\}
\end{aligned}
\tag{C.7}
$$

Since the cutoff function ρ_2 and the product of cutoff functions $\rho_2\rho_1^{-1}$ are all rapidly decreasing functions of k^2 in momentum Euclidean space, the contributions to the action (2.27) of the terms (C.5), (C.6) and (C.7) are readily seen to be all regularized.

In summary, all the contributions to the action described in this appendix are regularized, if the cutoff function entering the smeared field ϕ (2.32) behaves at least as (2.33), and if both the propagators of the gluons and the φ fields are rapidly decreasing functions in Euclidean momentum space in the IR and UV domains. Moreover the shift of momenta of the smeared field ϕ are allowed in all terms contributing to the action (2.27), apart in some terms of the expressions (C.1), (C.2) and (C.3).

Appendix D

After integration over the boson fields φ, only the proper diagrams with internal φ bosons lines must be considered. Regarding the structure of the action S_{Reg} (4.1) in momentum space, the vertices entering in such proper diagrams Γ can be classified in two sets.

Let $\Gamma^{1i}_{A\varphi}$ be the set of vertices given by the parts (C.1), (C.2), (C.3), (C.5), (C.6) and (C.7) of the action (4.1). In these vertices the derivative act in general on the boson fields φ. If N^{1i}_A is the number of gluon lines attached to the vertex $\Gamma^{1i}_{A\varphi}$, the order of the derivative acting on φ, is by inspection at most $4 - N^{1i}_A$. Notice that there are no derivatives acting on the φ fields in a set of vertices given by the parts (C.3) and (C.5) of the action (4.1). Thus if N^{1i}_φ is the number of φ fields attached to the vertex $\Gamma^{1i}_{A\varphi}$, by power counting the contribution of such a vertex to the UV behavior of the proper diagram Γ is

$$\Gamma^{1i}_{A\varphi} \sim \Lambda^{4-N^{1i}_A}(\Lambda^{-1})^{N^{1i}_\varphi}, \tag{D.1}$$

where Λ is the UV cutoff scale. The part S_π (3.12) of the action (4.1) gives the second set Γ^{2i}_φ of vertices. If N^{2i}_φ is the number of φ fields attached to the vertex Γ^{2i}_φ, the contribution of this vertex to the UV behavior of Γ is

$$\Gamma^{2i}_\varphi \sim M^2(\Lambda^{-1})^{N^{2i}_\varphi - 2}. \tag{D.2}$$

Now suppose that the proper vertex Γ is composed of L loops, with m vertices of type 1, n vertices of type 2 and I_φ internal lines. Since in the UV domain the propagators of the fields φ behave like M^{-2} (3.16), in this domain the proper vertex Γ scales like

$$\Gamma \sim \Lambda^{4L}\Lambda^{[4m-\sum_{i\in S_{1m}} N^{1i}_A - (\sum_{i\in S_{1m}} N^{1i}_\varphi + \sum_{i\in S_{2n}} N^{2i}_\varphi) + 2n]} M^{2n} M^{-2I_\varphi}, \tag{D.3}$$

where S_{1m} and S_{2n} are respectively a subset of m and n indices referring to the vertices of type 1 and 2. The total number V of vertices and the total number I_φ of internal lines

pertaining to the proper vertex Γ (D.3) are respectively given by

$$V = m + n, \tag{D.4}$$

and

$$I_\varphi = \tfrac{1}{2}\Big(\sum_{i\in S_{1m}} N_\varphi^{1i} + \sum_{i\in S_{2n}} N_\varphi^{2i} \Big). \tag{D.5}$$

Knowing that the number L of loops is related to V and I_φ [18] by

$$L = I_\varphi - V + 1, \tag{D.6}$$

the expression (D.3) can be reduced to

$$\Gamma \sim \big(\tfrac{\Lambda^2}{M^2}\big)^{I_\varphi - n} \Lambda^{-k}. \tag{D.7}$$

Here the integer k is given in terms of the total number $N_A = \sum_{i\in S_{1l}} N_A^{1i}$ of external gluon lines pertaining to the proper vertex Γ (D.3) by

$$k = N_A - 4, \tag{D.8}$$

and the dimension of Γ (D.7) is exactly that of Λ^{-k}. The values of k can be discussed according to the values of N_A.

If $N_A > 4$, (D.8) shows that k is strictly positive. Then, by the dimensional argument, when all the external momenta scale with Λ, the proper vertex Γ (D.7) is suppressed by a power of Λ. If $N_A \leq 4$, only the two, three and four effective gluons vertices must be considered. At a fixed loop order, these effective vertices are a sum of proper vertices given by (D.7). Therefore, from (D.7) and (D.8), the polarization operator, whose dimension is that of a mass squared, seems to behave at worst like $\big(\frac{\Lambda^2}{M^2}\big)^{I_\varphi - n}\Lambda^2$ in the UV domain. However from the dimensional ground, Lorentz invariance and gauge invariance of the theory, this tensorial operator is transverse and then behaves in the UV domain like $\big(\frac{\Lambda^2}{M^2}\big)^{I_\varphi}\mathcal{O}(\Lambda^0)$. This means that for the two gluons effective vertex, the minimal value of k is not -2, but zero. The behavior of the three gluons effective vertex, is at first sight $\big(\frac{\Lambda^2}{M^2}\big)^{I_\varphi}\Lambda$. Since the dimension of this vectorial operator is that of a mass, the Lorentz invariance imposes that this dimension is that of the external gluon momenta. In that case the minimal value of k is in fact zero. For the four gluons effective vertex under consideration, the minimal value of k (D.8) is zero.

References

[1] L.D. Faddeev and A.A. Slavnov. *Gauge Fields, Introduction to Quantum Theory*, (The Benjamin/Cummings Publishing Company, Reading, Massachussets, 1980).

[2] H.J Rothe, *Lattice Gauge theory:An Introduction*, Lectures Notes in Physics **59**,1, (Wordl Scientific, Singapore, 1997).

[3] See for instance J. Collins *Renormalization*, (Cambridge University Press, 1984).

[4] T. Becher and K. Melnikov, *Phys. Rev.* **D66**, 074508 (2002).

[5] S. Capitani, *Phys. Rep.* **382**, 113 (2003).

[6] T.D. Bakeyev and A.A. Slavnov, *Mod. Phys. Lett.* **A11**, 1539 (1996).

[7] V. Gribov, *Nucl. Phys.* **B139**, 1 (1978);
I. Singer, *Comm. Math. Phys.* **60**, 7 (1978).

[8] Z. Bern, M.B. Halpern, L. Sadun and C. Taubes, *Phys. Lett.* **B165**, 151 (1985);
Z. Bern, M.B. Halpern, L. Sadun and C. Taubes, *Nucl. Phys.* **B284**, 35 (1987).

[9] Z. Bern, M.B. Halpern and L. Sadun,, *Nucl. Phys.* **B284**, 92 (1987).

[10] C. Bagnuls and C. Bervillier, *Phys. Rep.* **348**, 91 (2001).

[11] J. Polchinski, *Nucl. Phys.* **B231**, 269 (1984).

[12] T.R. Morris and O.J. Rosten, *Phys. Rev.* **D73**, 065003 (2006).

[13] S. Arnone, Y.A. Kubyshin, T.R. Morris and J.F. Tighe, *Int. J. Mod. Phys.* **A17**, 2283 (2002).

[14] J. Zinn-Justin, *Quantum Field Theory and Critical Phenomena*, (Clarendon Press - Oxford, 1989).

[15] J. L. Jacquot, *Phys. Lett.* **B631**, 83 (2005).

[16] For a review on Stueckelberg fields, see H. Ruegg and M. Ruiz-Altaba, *Int. J. Mod. Phys.* **A19**, 3265 (2004).

[17] N. Dragon, T. Hurth and P. van Nieuwenhuizen, *Nucl. Phys. Proc. Suppl.* **56B**, 318 (1997).

[18] C. Itzykson and J.B. Zuber, *Quantum Field Theory*, (McGraw-Hill Inc.,New york, 1980).